The subtleties of randomness: An example of mislead intuition

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What is random and what is not can be a matter of very subtle debate. Not only is our notion of randomness often too vague, but also our "intuitive feel" for whether something is random or not can very easily be led astray. These notes provide a striking example.

A simple example of a random experiment would be to randomly place points inside the unit square. To be more precise from the beginning, what we mean by *random placement* is that (i) each point is placed *independently* from all the others and that (ii) there is a *constant probability per unit area* for the placement of points. This is just a two-dimensional Poisson-process.

However, here we want to demonstrate that *recognizing* a point distribution as originating from such a procedure is a whole different matter. Have a look at the two point distributions in Fig. 1. Both are random, but they appear to be different in their *evenness* in which points are placed. While in the upper picture there are quite noticeable "holes" and "clusters", the lower picture is more even — in some loose sense it shows less *structure*. For this reason most people would be inclined to believe that the lower picture corresponds to the random placement of points we initially talked about, while in the upper picture there must be some sort of "interactions" going on between the points which cause them to locally aggregate and thereby leave holes somewhere else.

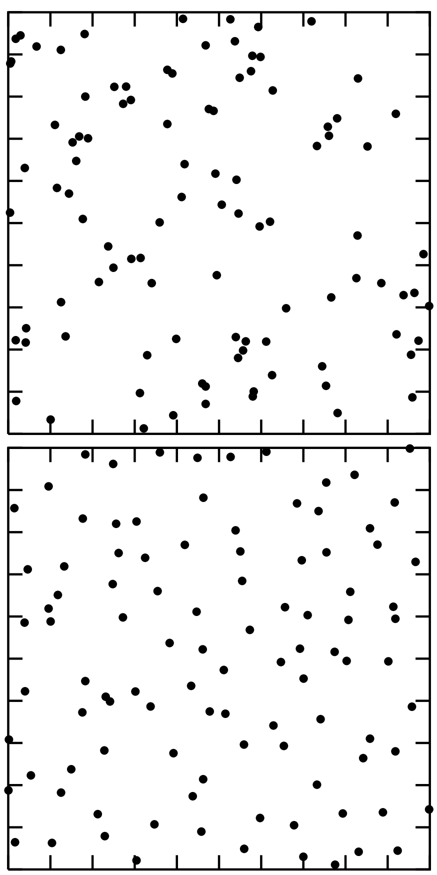

FIG. 1: Two different random placements of 100 points inside the unit square.

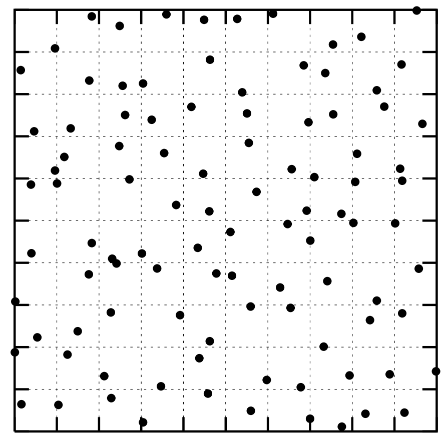

FIG. 2: Same as the lower picture in Fig. 1, but now plotted with additional grid-lines.

The surprising truth is that it is the other way around! The upper picture in Fig. 1 is completely random, while the lower picture is not. In fact, the lower picture is *too even*! This can be seen by drawing it again, but now placing a 10×10 grid over the square, as has been done in Fig. 2. We can quite readily see that there is *exactly one point within each sub-square*, therefore this distribution is most likely not the result of a completely random process: If it where, why is no sub-square empty, and why is none filled with two or more points? The probability of such evenness happening by pure chance in a completely random process is $1/\binom{199}{100} \approx 2.2 \times 10^{-59}$. Someone evidently has stacked the cards, and our intuition was almost 60 orders of magnitude off!

I've seen this striking example in 2001 in a conference talk on the ASTATPHYS-MEX in Cancun, but unfortunately I've forgotten who presented it.